COMPACTNESS FOR THE $\overline{\partial}$ - NEUMANN PROBLEM - A FUNCTIONAL ANALYSIS APPROACH.

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Abstract.
We discuss compactness of the $\overline{\partial}$-Neumann operator in the setting of weighted L^2-spaces on $\mathbb{C}^n$. For this purpose we use a description of relatively compact subsets of L^2- spaces. We also point out how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain and mention an abstract functional analysis characterization of compactness of the $\overline{\partial}$-Neumann operator.

1. Introduction.

In this paper we continue the investigations of [HaHe] concerning existence and compactness of the canonical solution operator to $\overline{\partial}$ on weighted L^2-spaces over $\mathbb{C}^n$.
Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function and define the space

$$L^2(\mathbb{C}^n, \varphi) = \{f : \mathbb{C}^n \longrightarrow \mathbb{C} \ : \ \int_{\mathbb{C}^n} |f|^2 \, e^{-\varphi} \, d\lambda < \infty\},$$

where λ denotes the Lebesgue measure, the space $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ of $(0,1)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$ and the space $L^2_{(0,2)}(\mathbb{C}^n, \varphi)$ of $(0,2)$-forms with coefficients in $L^2(\mathbb{C}^n, \varphi)$. Let

$$\langle f, g\rangle_\varphi = \int_{\mathbb{C}^n} f \, \overline{g} e^{-\varphi} \, d\lambda$$

denote the inner product and

$$\|f\|^2_\varphi = \int_{\mathbb{C}^n} |f|^2 e^{-\varphi} \, d\lambda$$

the norm in $L^2(\mathbb{C}^n, \varphi)$.
We consider the weighted $\overline{\partial}$-complex

$$L^2(\mathbb{C}^n, \varphi) \underset{\overleftarrow{\overline{\partial}^*_\varphi}}{\xrightarrow{\overline{\partial}}} L^2_{(0,1)}(\mathbb{C}^n, \varphi) \underset{\overleftarrow{\overline{\partial}^*_\varphi}}{\xrightarrow{\overline{\partial}}} L^2_{(0,2)}(\mathbb{C}^n, \varphi),$$

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where $\overline{\partial}^*_\varphi$ is the adjoint operator to $\overline{\partial}$ with respect to the weighted inner product. For $u = \sum_{j=1}^n u_j d\overline{z}_j \in \mathrm{dom}(\overline{\partial}^*_\varphi)$ one has

$$\overline{\partial}^*_\varphi u = -\sum_{j=1}^n \left(\frac{\partial}{\partial z_j} - \frac{\partial \varphi}{\partial z_j} \right) u_j .$$

The complex Laplacian on $(0,1)$-forms is defined as

$$\Box_\varphi := \overline{\partial}\, \overline{\partial}^*_\varphi + \overline{\partial}^*_\varphi \overline{\partial},$$

where the symbol $\Box_\varphi$ is to be understood as the maximal closure of the operator initially defined on forms with coefficients in $\mathcal{C}_0^\infty$, i.e., the space of smooth functions with compact support.
$\Box_\varphi$ is a selfadjoint and positive operator, which means that

$$\langle \Box_\varphi f, f \rangle_\varphi \geq 0 \ , \ \text{for } f \in dom(\Box_\varphi).$$

The associated Dirichlet form is denoted by

$$Q_\varphi(f,g) = \langle \overline{\partial} f, \overline{\partial} g \rangle_\varphi + \langle \overline{\partial}^*_\varphi f, \overline{\partial}^*_\varphi g \rangle_\varphi, \tag{1.1}$$

for $f, g \in dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$. The weighted $\overline{\partial}$-Neumann operator N_φ is - if it exists - the bounded inverse of $\Box_\varphi$.

We indicate that $f \in dom(\overline{\partial}^*_\varphi)$ if and only if

$$\sum_{j=1}^n \left(\frac{\partial f_j}{\partial z_j} - \frac{\partial \varphi}{\partial z_j}\, f_j \right) \in L^2(\mathbb{C}^n, \varphi)$$

and that forms with coefficients in $\mathcal{C}_0^\infty(\mathbb{C}^n)$ are dense in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|_\varphi^2 + \|\overline{\partial}^*_\varphi f\|_\varphi^2)^{\frac{1}{2}}$ (see [GaHa]).

Now we suppose that the lowest eigenvalue μ_φ of the Levi - matrix

$$M_\varphi = \left(\frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k} \right)_{jk}$$

of φ satisfies

$$\liminf_{|z| \to \infty} \mu_\varphi(z) > 0, \quad (*)$$

and mention the Kohn-Morrey formula:

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2 = \sum_{j,k=1}^n \int_{\mathbb{C}^n} \left| \frac{\partial u_j}{\partial \overline{z}_k} \right|^2 e^{-\varphi}\, d\lambda + \int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda \tag{1.2}$$

from which we get

$$\int_{\mathbb{C}^n} \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}\, u_j \overline{u}_k\, e^{-\varphi}\, d\lambda \leq \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2, \tag{1.3}$$

hence for a plurisubharmonic weight function φ satisfying (*), there is a $C > 0$ such that

$$\|u\|_\varphi^2 \leq C(\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2)$$

for each $(0,1)$-form $u \in$dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$.
For the proof see [FS], [GaHa] or [Str].
Now it follows that there exists a uniquely determined $(0,1)$-form $N_\varphi u \in$dom $(\overline{\partial}) \cap$ dom $(\overline{\partial}^*_\varphi)$ such that

$$\langle u, v\rangle_\varphi = Q_\varphi(N_\varphi u, v) = \langle \overline{\partial} N_\varphi u, \overline{\partial} v\rangle_\varphi + \langle \overline{\partial}^*_\varphi N_\varphi u, \overline{\partial}^*_\varphi v\rangle_\varphi,$$

and that

$$\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}^*_\varphi N_\varphi u\|_\varphi^2 \leq C_1 \|u\|_\varphi^2 \tag{1.4}$$

which means that

$$N_{1,\varphi} : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \text{dom}\,(\overline{\partial}) \cap \text{dom}\,(\overline{\partial}^*_\varphi)$$

is continuous in the graph topology, as well as

$$\|N_\varphi u\|_\varphi^2 \leq C_2(\|\overline{\partial} N_\varphi u\|_\varphi^2 + \|\overline{\partial}^*_\varphi N_\varphi u\|_\varphi^2) \leq C_3 \|u\|_\varphi^2,$$

where $C_1, C_2, C_3 > 0$ are constants. Hence we get that N_φ is a continuous linear operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself (see also [ChSh]).
We will give a new proof of the main result in [HaHe] using a direct approach, see [B], Corollaire IV.26, where two conditions are given which imply that a subset of an L^2-space is relatively compact. The first of these conditions will correspond to Gårding's inequality (see for instance [F] , [GaHa],) and the second condition corresponds to our assumption on the lowest eigenvalue of the Levi matrix M_φ.
We indicate how to use this method to show that property (P) implies compactness for the $\overline{\partial}$-Neumann operator on a smoothly bounded pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ and finally mention an abstract necessary and sufficient condition for the $\overline{\partial}$- Neumann operator to be compact.

2. Weighted Sobolev spaces

Now we define an appropriate Sobolev space and prove compactness of the corresponding embedding, for related settings see [BDH], [Jo], [KM] .

Definition 2.1. *Let*

$$\mathcal{W}^{Q_\varphi} = \{u \in L^2_{(0,1)}(\mathbb{C}^n, \varphi) \ : \ \|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2 < \infty\}$$

with norm

$$\|u\|_{Q_\varphi} = (\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}^*_\varphi u\|_\varphi^2)^{1/2}.$$

Remark: $\mathcal{W}^{Q_\varphi}$ coincides with the form domain $dom(\overline{\partial}) \cap dom(\overline{\partial}^*_\varphi)$ of Q_φ (see [Ga], [GaHa]).

Proposition 2.2. *Suppose that the weight function φ is plurisubharmonic and that the lowest eigenvalue μ_φ of the Levi - matrix M_φ satisfies*

$$\lim_{|z|\to\infty} \mu_\varphi(z) = +\infty\,. \quad (^{**})$$

Then the embedding

$$j_\varphi : \mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

is compact.

Proof. For $u \in \mathcal{W}^{Q_\varphi}$ we have by 1.3

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq \langle M_\varphi u, u\rangle_\varphi.$$

This implies

$$\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi \geq \int_{\mathbb{C}^n} \mu_\varphi(z)\, |u(z)|^2\, e^{-\varphi(z)}\, d\lambda(z). \tag{2.1}$$

We show that the unit ball in $\mathcal{W}^{Q_\varphi}$ is relatively compact in $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$. For this purpose we use the following lemma, see for instance [B] Corollaire IV.26.

Lemma 2.3. *Let $\mathcal{A}$ be a bounded subset of $L^2(\mathbb{C}^n, \varphi)$. Suppose that*
(i) for each $\epsilon > 0$ and for each $R > 0$ there exists $\delta > 0$ such that

$$\|\tau_h f - f\|_{L^2(\mathbb{B}_R, \varphi)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in \mathcal{A}$, where $\tau_h f(z) = f(z+h)$ and $\mathbb{B}_R = \{z \in \mathbb{C}^n : |z| < R\}$;
(ii) for each $\epsilon > 0$ there exists $R > 0$ such that

$$\|f\|_{L^2(\mathbb{C}^n \setminus \mathbb{B}_R, \varphi)} < \epsilon$$

for each $f \in \mathcal{A}$.
Then $\mathcal{A}$ is relatively compact in $L^2(\mathbb{C}^n, \varphi)$.

Remark 2.4. *Conditions (i) and (ii) are also necessary for $\mathcal{A}$ to be relatively compact in $L^2(\mathbb{C}^n, \varphi)$ (see* [B]*).*

First we show that condition (i) of Lemma 2.3 is satisfied in our situation. Let $u = \sum_{j=1}^n u_j\, dz_j$ be a $(0,1)$-form with coefficients in $\mathcal{C}^\infty_0$. For each u_j and for $t \in \mathbb{R}$ and $h = (h_1, \dots, h_n) \in \mathbb{C}^n$ let

$$v_j(t) := u_j(z + th).$$

Note that

$$|v_j'(t)| \leq |h| \left[\sum_{k=1}^n \left(\left| \frac{\partial u_j}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial u_j}{\partial y_k}(z+th) \right|^2 \right) \right]^{1/2},$$

where $z_k = x_k + iy_k$, for $k = 1, \dots, n$. By the fact that

$$u_j(z+h) - u_j(z) = v_j(1) - v_j(0) = \int_0^1 v_j'(t)\, dt$$

we can now estimate for $|h| < R$

$$\int_{\mathbb{B}_R} |\tau_h u_j(z) - u_j(z)|^2 e^{-\varphi(z)}\, d\lambda(z) = \int_{\mathbb{B}_R} |\tau_h(\chi_R u_j)(z) - \chi_R u_j(z)|^2 e^{-\varphi(z)}\, d\lambda(z)$$
$$\leq |h|^2 \int_{\mathbb{B}_R} \left[\int_0^1 \sum_{k=1}^n \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z+th) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z+th) \right|^2 \right) dt \right] e^{-\varphi(z)}\, d\lambda(z)$$
$$\leq C_{R,\varphi}\, |h|^2 \int_{\mathbb{B}_{3R}} \sum_{k=1}^n \left(\left| \frac{\partial(\chi_R u_j)}{\partial x_k}(z) \right|^2 + \left| \frac{\partial(\chi_R u_j)}{\partial y_k}(z) \right|^2 \right) e^{-\varphi(z)}\, d\lambda(z)$$

for $j = 1, \dots, n$ where χ_R is a $\mathcal{C}^\infty$ cutoff function which is identically 1 on $\mathbb{B}_{2R}$ and zero outside $\mathbb{B}_{3R}$ and by Gårding's inequality for $\mathbb{B}_{3R}$ (see [ChSh], [F], [GaHa])

$$\begin{aligned} \|\chi_R u\|^2_{\varphi,1} &\leq C'_{\varphi,R} \left(\|\overline{\partial}(\chi_R u)\|^2_\varphi + \|\overline{\partial}^*_\varphi(\chi_R u)\|^2_\varphi + \|\chi_R u\|^2_\varphi \right) \\ &\leq C''_{\varphi,R} \left(\|\overline{\partial} u\|^2_\varphi + \|\overline{\partial}^*_\varphi u\|^2_\varphi + \|u\|^2_\varphi \right) \end{aligned}$$

we can control the last integral by the norm $\|u\|^2_{Q_\varphi}$. Since we started from the unit ball in $\mathcal{W}^{Q_\varphi}$ we get that condition (i) of Lemma 2.3 is satisfied.

Condition (ii) of Lemma 2.3 is satisfied for the unit ball of $\mathcal{W}^{Q_\varphi}$ since we have

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \leq \int_{\mathbb{C}^n \setminus \mathbb{B}_R} \frac{\mu_\varphi(z)|u_(z)|^2}{\inf\{\mu_\varphi(z) : |z| \geq R\}} e^{-\varphi(z)} d\lambda(z).$$

So formula (2.1) together with assumption (**) shows that

$$\int_{\mathbb{C}^n \setminus \mathbb{B}_R} |u(z)|^2 e^{-\varphi(z)}\, d\lambda(z) \leq \frac{\|u\|^2_{Q_\varphi}}{\inf\{\mu_\varphi(z)\, :\, |z| \geq R\}} < \epsilon,$$

if R is big enough.

□

We are now able to give a short proof of the main result in [HaHe] or [GaHa]

Proposition 2.5. *Let φ be a plurisubharmonic $\mathcal{C}^2$- weight function. If the lowest eigenvalue $\mu_\varphi(z)$ of the Levi - matrix M_φ satisfies (**), then N_φ is compact.*

Proof. By Proposition 2.2, the embedding $\mathcal{W}^{Q_\varphi} \hookrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ is compact. The inverse N_φ of $\Box_\varphi$ is continuous as an operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into $\mathcal{W}^{Q_\varphi}$, this follows from 1.4. Therefore we have that N_φ is compact as an operator from $L^2_{(0,1)}(\mathbb{C}^n, \varphi)$ into itself. □

Now notice that

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

can be written in the form

$$N_\varphi = j_\varphi \circ j_\varphi^*\ ,$$

where

$$j_\varphi^* : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow \mathcal{W}^{Q_\varphi}$$

is the adjoint operator to j_φ (see [Str]).
This means that N_φ is compact if and only if j_φ is compact and summarizing the above results we get the following

Proposition 2.6. *Let $\varphi : \mathbb{C}^n \longrightarrow \mathbb{R}^+$ be a plurisubharmonic $\mathcal{C}^2$-weight function . The $\overline{\partial}$-Neumann operator*

$$N_\varphi : L^2_{(0,1)}(\mathbb{C}^n, \varphi) \longrightarrow L^2_{(0,1)}(\mathbb{C}^n, \varphi)$$

is compact if and only if for each $\epsilon > 0$ there exists $R > 0$ such that

$$\|u\|_{L^2_{(0,1)}(\mathbb{C}^n \setminus \mathbb{B}_R, \varphi)} < \epsilon$$

for each $u \in \mathcal{W}^{Q_\varphi}$ with

$$\|\overline{\partial} u\|_\varphi^2 + \|\overline{\partial}_\varphi^* u\|_\varphi^2 \leq 1.$$

3. Smoothly bounded pseudoconvex domains and properties (P) and ($\tilde{\mathrm{P}}$)

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Ω satisfies property (P), if or each $M > 0$ there exists a a neighborhood U of $\partial\Omega$ and a plurisubharmonic function $\varphi_M \in \mathcal{C}^2(U)$ such that

$$\sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(p) t_j \overline{t}_k \geq M\|t\|^2,$$

for all $p \in \partial\Omega$ and for all $t \in \mathbb{C}^n$.

Ω satisfies property ($\tilde{\mathrm{P}}$) if the following holds: there is a constant C such that for all $M > 0$ there exists a $\mathcal{C}^2$ function φ_M in a neighborhood U (depending on M) of $\partial\Omega$ with
(i) $\left|\sum_{j=1}^n \frac{\partial \varphi_M}{\partial z_j}(z) t_j\right|^2 \leq C \sum_{j=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k$
and
(ii) $\sum_{j=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k}(z) t_j \overline{t}_k \geq M\|t\|^2$,
for all $z \in U$ and for all $t \in \mathbb{C}^n$.

In [C] Catlin showed that condition (P) implies compactness of the $\overline{\partial}$- operator N on $L^2_{(0,1)}(\Omega)$ and McNeal ([McN]) showed that property ($\tilde{\mathrm{P}}$) also implies compactness of the $\overline{\partial}$- operator N on $L^2_{(0,1)}(\Omega)$. It is not difficult to show that property (P) implies property ($\tilde{\mathrm{P}}$), see for instance [Str].
We can now use a similar approach as in section 2 to prove Catlin's result. For this purpose we use the following version of lemma 2.3

Lemma 3.1. *Let $\mathcal{A}$ be a bounded subset of $L^2(\Omega)$. Suppose that*
(i) for each $\epsilon > 0$ and for each $\omega \subset\subset \Omega$ there exists $\delta > 0, \delta < dist(\omega, \Omega^c)$ such that

$$\|\tau_h f - f\|_{L^2(\omega)} < \epsilon$$

for each $h \in \mathbb{C}^n$ with $|h| < \delta$ and for each $f \in \mathcal{A}$,
(ii) for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\|f\|_{L^2(\Omega\setminus\omega)} < \epsilon$$

for each $f \in \mathcal{A}$.
Then $\mathcal{A}$ is relatively compact in $L^2(\Omega)$.

Remark 3.2. *Conditions (i) and (ii) are also necessary for $\mathcal{A}$ to be relatively compact in $L^2(\Omega)$.*

In order to show that the unit ball in $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ satisfies condition (i) of 3.1 we remark that Gårding's inequality holds for $\omega \subset\subset \Omega$ (see section 2). To verify condition (ii) we use property (P) and the following version of the Kohn-Morrey formula

$$\int_\Omega \sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k \, e^{-\varphi_M} \, d\lambda \leq \|\overline{\partial} u\|^2_{\varphi_M} + \|\overline{\partial}^*_{\varphi_M} u\|^2_{\varphi_M}, \tag{3.1}$$

here we used that Ω is pseudoconvex, which means that the boundary terms in the Kohn-Morrey formula can be neglected. Now we point out that the weighted $\overline{\partial}$- complex is equivalent to the unweighted one and that the expression $\sum_{j=1}^n \frac{\partial \varphi_M}{\partial z_j} u_j$ which appears in $\overline{\partial}^*_{\varphi_M} u$, can be controlled by the complex Hessian $\sum_{j,k=1}^n \frac{\partial^2 \varphi_M}{\partial z_j \partial \overline{z}_k} u_j \overline{u}_k$, which follows from the fact that property (P) implies property ($\tilde{\mathrm{P}}$) (see [Str]). Of course we also use that the weight φ_M is bounded on $\Omega \subset\subset \mathbb{C}^n$. In this way the same reasoning as in section 2 shows that property (P) implies condition (ii) of lemma 3.1. Therefore condition (P) gives that the unit ball of $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$ is relatively compact in $L^2_{(0,1)}(\Omega)$ and hence that the $\overline{\partial}$-Neumann operator is compact.

Now let

$$j : dom(\overline{\partial}) \cap dom(\overline{\partial}^*) \hookrightarrow L^2_{(0,1)}(\Omega)$$

denote the embedding. It follows from [Str] that

$$N = j \circ j^*.$$

Hence N is compact if and only if j is compact, where $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ is endowed with the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$.

Proposition 3.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded pseudoconvex domain. Let $\mathcal{B}$ denote the unit ball of $dom(\overline{\partial}) \cap dom(\overline{\partial}^*)$ in the graph norm $f \mapsto (\|\overline{\partial} f\|^2 + \|\overline{\partial}^* f\|^2)^{\frac{1}{2}}$. The $\overline{\partial}$- Neumann operator N is compact if and only if $\mathcal{B}$ as a subset of $L^2_{(0,1)}(\Omega)$ satisfies the following condition:*
for each $\epsilon > 0$ there exists $\omega \subset\subset \Omega$ such that

$$\|f\|_{L^2_{(0,1)}(\Omega\setminus\omega)} < \epsilon$$

for each $f \in \mathcal{B}$.

This follows from the above remarks about the embedding j and the fact that the two conditions in 3.1 are also necessary for a bounded set in L^2 to be relatively compact. For a localized version of the above result see [Sa].

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